EIGENVALUES OF THE LAPLACIAN AND EXTRINSIC GEOMETRY

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Abstract. We extend the results given by Colbois, Dryden and El Soufi on the relationships between the eigenvalues of the Laplacian and an extrinsic invariant called *intersection index*, in two directions. First, we replace this intersection index by invariants of the same nature which are stable under *small perturbations*. Second, we concern complex submanifolds of the complex projective space $\mathbb{C}P^N$ instead of submanifolds of $\mathbb{R}^N$. We obtain an upper bound depending only on the degree of the submanifold and which is sharp for the first non-zero eigenvalue.

1. Introduction and statement of the results

The investigation of the relationships between the extrinsic geometry of submanifolds and the spectrum of the Laplace-Beltrami operator is an important topic of spectral geometry. The purpose of this paper, inspired by the recent work of Colbois, Dryden and El Soufi, is to give upper bounds for the eigenvalues of the Laplacian in terms of some extrinsic data. The Laplace-Beltrami operator Δ consists a nondecreasing sequence of positive real numbers

$$0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \nearrow \infty.$$

Colbois, Dryden and El Soufi [6] study the relation between an extrinsic invariant of submanifolds of $\mathbb{R}^N$, called the *intersection index*, and the eigenvalues of the Laplace-Beltrami operator. For a compact m-dimensional immersed submanifold M of $\mathbb{R}^N = \mathbb{R}^{m+p}$, $p > 0$, the *intersection index* is given by

$$i(M) = \sup_{\Pi} \sharp(M \cap \Pi),$$

where Π runs over the set of all p-planes that are transverse to M ; if M is not embedded, we count multiple points of M according to their multiplicity. We remark that the intersection index was also investigated by Thom [3] where it was called the *degree* of M.

In [6], Colbois, Dryden and El Soufi show that there is a positive constant

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$c(m)$, depending only on m, such that for every compact m-dimensional immersed submanifold of $\mathbb{R}^{m+p}$, we have the following inequality

$$\lambda_k(M)\mathrm{Vol}(M)^{2/m} \leq c(m) i(M)^{2/m} k^{2/m}. \tag{1}$$

Moreover, the intersection index in the above inequality is not replaceable with a constant depending only on the dimension m. Even for hypersurfaces, the first positive eigenvalue cannot be controlled only in terms of the volume and the dimension (see [6, Theorem 1.4]).
An immediate consequence of Inequality (1) is the fact that for convex hypersurfaces, the normalized eigenvalues are bounded above only in terms of the dimension. Another remarkable consequence of Inequality (1) concerns algebraic submanifolds [6, Corollary 4.1]: Let M be a compact real algebraic manifold, i.e. M is a zero locus of p real polynomials in $m+p$ variables of degrees $N_1, \dots, N_p$. Then

$$\lambda_k(M)\mathrm{Vol}(M)^{2/m} \leq c(m) N_1^{2/m} \cdots N_p^{2/m} k^{2/m}. \tag{2}$$

Note that Inequality (1) is not stable under "*small*" perturbations, since the intersection index might dramatically change.

In this paper, we extend the work of Colbois, Dryden and El Soufi in two directions. The first one consists in replacing the intersection index $i(M)$ by invariants of the same nature which are stable under *small* perturbations. The second direction concerns complex submanifolds of the complex projective space $\mathbb{C}P^N$.

First part. Let $\varepsilon < 1$ be a positive number. By a *ε-small perturbation*, we mean any perturbation in a region $D \subset M$ whose measure is at most equal to εVol(M). To avoid any technical complexity, we assume that $M \setminus D$ is a smooth manifold with smooth boundary. Here, we define new notions of intersection indexes which are stable under any ε-smal perturbation. Let G be the Grassmannian of all m-vector spaces in $\mathbb{R}^{m+p}$ endowed with an invariant Haar measure with total measure 1. Let $0 < \varepsilon < 1$ and D be any open subdomain of M such that $M \setminus D$ is a smooth manifold with smooth boundary and $\mathrm{Vol}(D) \leq \varepsilon \mathrm{Vol}(M)$. We denote $M \setminus D$ by M_ε^D and $\sup_{P \perp H} \sharp(M_\varepsilon^D \cap P)$ by $i_H(M_\varepsilon^D)$, where P is an affine p-plane orthogonal to H. We now define the *ε-mean intersection index* as follows:

$$\bar{\imath}^\varepsilon(M) := \inf_D \int_G i_H(M_\varepsilon^D) dH,$$

where D runs over regions whose measure is smaller than $\varepsilon\mathrm{Vol}(M)$ and $M \setminus D$ is a smooth manifold with smooth boundary.
Similarly, for $r > 0$, we define the *(r,ε)-local intersection index* as:

$$\bar{\imath}_r^\varepsilon(M) := \inf_D \sup_{x \in M_\varepsilon^D} \int_G i_H(M_\varepsilon^D \cap B(x,r)) dH,$$

where $B(x,r) \subset \mathbb{R}^{m+p}$ is an Euclidean ball centered at x and of radius r and D runs over regions whose measure is smaller than $\varepsilon\mathrm{Vol}(M)$ and $M \setminus D$

is a smooth manifold with smooth boundary.
We can now state our theorem.

Theorem 1.1. *There exist positive constants c_m, α_m and β_m depending only on m such that for every compact m-dimensional immersed submanifold M of $\mathbb{R}^{m+p}$ and every $k \in N^*$ and $\varepsilon > 0$, we have*

$$\lambda_k(M)\, Vol(M)^{2/m} \leq c_m \frac{\overline{\imath}^{\varepsilon}(M)^{2/m}}{(1-\varepsilon)^{1+2/m}} k^{2/m}, \tag{3}$$

and

$$\lambda_k(M) \leq \alpha_m \frac{1}{(1-\varepsilon)r^2} + \beta_m \frac{\overline{\imath}_r^{\varepsilon}(M)^{2/m}}{(1-\varepsilon)^{1+2/m}} \left(\frac{k}{Vol(M)}\right)^{2/m}. \tag{4}$$

The main feature of the inequalities of Theorem 1.1 is the fact that the upper bounds are not considerably affected by the presence of a large intersection index in a "small" part of M (i.e. a subdomain with small volume). In particular, for a compact hypersurface of $\mathbb{R}^{m+1}$ which is convex outside a region[1] D of measure at most εVol(M), one has $\overline{\imath}^{\varepsilon}(M) \leq i(M_D^{\varepsilon})$ and then

$$\lambda_k(M)\mathrm{Vol}(M)^{2/m} \leq c_m \frac{2^{2/m}}{(1-\varepsilon)^{1+2/m}} k^{2/m}.$$

We note that one also has Inequality (2) when M is a algebraic polynomial outside of a region D of measure at most εVol(M).

Second part. We study another natural context where algebraic submanifolds can be considered which is the complex projective space $\mathbb{C}P^N$. According to Chow's Theorem ([10]), every complex submanifold M of $\mathbb{C}P^N$ is a smooth algebraic variety, i.e. it is a zero locus of a family of complex polynomials. We obtain the following upper bound for complex submanifolds of $\mathbb{C}P^N$ endowed with Fubini-Study metric g_{FS}.

Theorem 1.2. *Let M^m be an m-dimensional complex manifold admitting a holomorphic immersion $\phi : M \to \mathbb{C}P^N$. Then for every $k \in \mathbb{N}^*$ we have*

$$\lambda_{k+1}(M, \phi^* g_{FS}) \leq 2(m+1)(m+2)k^{\frac{1}{m}} - 2m(m+1). \tag{5}$$

In particular, one has Inequality (5) for every complex submanifold of $\mathbb{C}P^N$. Note that the power of k is compatible with the Weyl law.
Under the assumption of Theorem 1.2, for $k = 1$, one has

$$\lambda_2(M, \phi^* g_{FS}) \leq 4(m+1), \tag{6}$$

which is a sharp inequality since the equality holds for $\mathbb{C}P^m$. This sharp upper bound was also obtained by Bourguignon, Li and Yau [2, page 200] (see also the paper by Arezzo, Ghigi and Loi [1]). Theorem 1.2 gives us another proof of this sharp inequality ; moreover, we do not need any restriction on holomorphic immersions (see page 11).

[1] We say that M is convex outside of D if after a perturbation of M which is the identity outside of D we get a convex compact hypersurface.

For a complex submanifold M of $\mathbb{C}P^{m+p}$ of the complex dimension m, we have

$$\mathrm{Vol(M)} = \deg(M)\mathrm{Vol}(\mathbb{C}P^m),$$

where $\deg(M)$ is the intersection number of M with a projective p-plane in a generic position. Moreover, one can describe M as a zero locus of a family of irreducible homogenous polynomials and then $\deg(M)$ is the multiplication of degrees of the irreducible polynomials that describe M (see for example [10, pages 171-172]). Therefore, we can rewrite Inequality (5). Since

$$\lambda_{k+1}(M, g_{FS})\mathrm{Vol}(M)^{\frac{1}{m}} = \lambda_{k+1}(M, g_{FS})\left(\deg(M)\mathrm{Vol}(\mathbb{C}P^m)\right)^{\frac{1}{m}},$$

hence,

$$\lambda_{k+1}(M, g_{FS})\mathrm{Vol}(M)^{\frac{1}{m}} \leq C(m)\deg(M)^{\frac{1}{m}}k^{\frac{1}{m}}. \tag{7}$$

One can now compare Inequality (7) with Inequality (2).
This paper is organized as follows: In section 2, we present a more general and abstract setting and we illustrate its applications in Section 3 where we prove Theorem 1.1. In Section 4, we consider algebraic submanifolds of $\mathbb{C}P^N$ and we prove Theorem 1.2. The method which is used in Section 4 to show Theorem 1.2 is independent from what we introduce in Sections 2 and 3.

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2. A general preliminary result

This section is devoted to introduce a more general and abstract setting. It is an illustration of the metric construction introduced by Colbois and Maerten [8]. They introduced a metric approach to construct an elaborated family of disjoint domains in a metric-measure space ($m-m$ space) X.
Throughout this section the triple (X, d, μ) will designate a complete locally compact $m-m$ space with a distance d and a finite and positive non-atomic Borel measure μ. We also assume that balls in (X, d) are pre-compact. Each pair (F, G) of Borel sets in X such that $F \subset G$ is called a capacitor. For $F \subseteq X$ and $r > 0$, we denote the *r-neighborhood of* F by F^r, that is

$$F^r = \{x \in X : d(x, F) \leq r\}.$$

Definition 2.1. *Given $\kappa > 1$, $\rho > 0$ and $N \in \mathbb{N}^*$, we say that a metric space (X, d) satisfies the $(\kappa, N; \rho)$-covering property if each ball of radius $0 < r \leq \rho$ can be covered by N balls of radius $\frac{r}{\kappa}$.*

Lemma 2.1. ([8, Corollary 2.3] and [7, Lemma 2.1]) *Let (X, d, μ) be an $m-m$ space satisfying the $(4, N; \rho)$-covering property. For every $n \in \mathbb{N}^*$, let $0 < r \leq \rho$ be such that for each $x \in X$, $\mu(B(x, r)) \leq \frac{\mu(X)}{4N^2 n}$. Then there exists a family $\mathcal{A} = \{(A_i, A_i^r)\}_{i=1}^n$ of capacitors in X such that*

(a) for each i, $\mu(A_i) \geq \frac{\mu(X)}{2Nn}$, and
(b) the subsets $\{A_i^r\}_{i=1}^n$ are mutually disjoint.

We define the *dilatation* of a function $f : (X, d) \to \mathbb{R}$ as

$$\mathrm{dil}(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)},$$

and the *local dilatation* at $x \in X$ as

$$\mathrm{dil}_x(f) = \lim_{\varepsilon \to 0} \mathrm{dil}(f|_{B(x,\varepsilon)}).$$

When different distance functions are considered, $\mathrm{dil}_d(f)$ and $\mathrm{dil}_{d,x}(f)$ stand for the dilatation and local dilatation at x associated with the distance d.
A map f is called Lipschitz if $\mathrm{dil}(f) < \infty$. Let (M, g) be a Riemannian manifold and d_g be the distance associated to the Riemannian metric g. A Lipschitz function on a Riemannian manifold M is differentiable almost everywhere and $|\nabla_g f(x)|$ coincides with $\mathrm{dil}_x(f)$ almost everywhere. Hence, $|\nabla_g f(x)| \leq \mathrm{dil}(f)$ almost everywhere.
Given a capacitor (F, G), let $\mathcal{T}(F, G)$ be the set of all compactly supported real valued functions on X so that for every $\varphi \in \mathcal{T}(F, G)$ we have supp $\varphi \subset \mathrm{G}^\circ = \mathrm{G} \setminus \partial \mathrm{G}$ and $\varphi \equiv 1$ in a neighborhood of F.
The following theorem relies on the construction given in the above lemma. It gives a construction of a family of disjointly supported functions with a nice control on their dilatations provided some conditions on the metric-measure structure are imposed. We will see its application in Corollary 2.1.

Theorem 2.1. *Let positive constants p, ρ, L and N be given and (X, d, μ) be an $m-m$ space satisfying the $(4, N; \rho)$-covering property and*

$$\mu(B(x, r)) \leq Lr^p, \quad \text{for every } x \in X \text{ and } 0 < r \leq \rho.$$

Then for every $n \in \mathbb{N}^*$ and every $r \leq \min\{\rho, \left(\frac{\mu(X)}{4N^2 Ln}\right)^{1/p}\}$ there is a family of n mutually disjoint bounded capacitors $\{(A_i, A_i^r)\}_{i=1}^n$, of X and a family $\{f_i\}$ of n Lipschitz functions with $f_i \in \mathcal{T}(A_i, A_i^r)$ such that $\mu(A_i) \geq \frac{\mu(X)}{2Nn}$ and

$$\mathrm{dil}_d(f_i) \leq \frac{1}{\rho} + (4N^2 L)^{1/p} \left(\frac{n}{\mu(X)}\right)^{1/p}. \tag{8}$$

If the condition $\mu(B(x, r)) \leq Lr^p$ is satisfied for every $r > 0$ then we take $\rho = \infty$. Hence, the first term at the right-hand side of the above inequality vanishes.

Proof of Theorem 2.1. According to Colbois and Maerten's result (see Lemma 2.1), if the $m-m$ space (X,d,μ) has $(4,N;\rho)$-covering property, then for every $r\leq\rho$ such that

$$\mu(B(x,r))\leq\frac{\mu(X)}{4N^2n},\quad\forall x\in X,\tag{9}$$

we have a family $\{(A_i,A_i^r)\}$ of mutually disjoint capacitors of X with the desired property mentioned in the theorem. We claim that when $r\leq\min\{\rho,\left(\frac{\mu(X)}{4N^2Ln}\right)^{1/p}\}$, the Inequality (9) is automatically satisfied. Indeed, according to the assumptions we have

$$\mu(B(x,r))\leq Lr^p\leq\min\{L\rho^p,\frac{\mu(X)}{4N^2n}\}\leq\frac{\mu(X)}{4N^2n}.$$

We now consider Lipschitz functions f_i's supported on A_i^r with $f_i(x)=1-\frac{d(x,A_i)}{r}$ on $A_i^r\setminus A_i$, $f_i(x)=1$ on A_i and zero outside of A_i^r. One can easily check that $\mathrm{dil}_d(f_i)\leq\frac{1}{r}$. Hence, we obtain:

$$\mathrm{dil}_d(f_i)\leq\frac{1}{\rho}+(4N^2L)^{1/p}\left(\frac{k}{\mu(X)}\right)^{1/p}.$$

This completes the proof. □

Let (M,g,μ) be a Riemannian manifold endowed with a finite non-atomic Borel measure μ. We define the following quantity that coincides with the eigenvalues of the Laplace-Beltrami operator when μ coincides with the Riemannian measure μ_g.

$$\lambda_k(M,g,\mu):=\inf_L\sup\{R(f):f\in L\},$$

where L is a k-dimensional vector space of Lipschitz functions and

$$R(f)=\frac{\int_M|\nabla_g f|^2d\mu}{\int_M f^2d\mu}$$

As an application of Theorem 2.1 in the Riemannian case, we have

Corollary 2.1. *Let (M,g,μ) be a Riemannian manifold with a finite non-atomic Borel measure μ and the distance d_g associated to the Riemannian metric g. If there exists a measure ν and a distance d so that*

$$d(x,y)\leq d_g(x,y),\quad\forall x,y\in M;\tag{10}$$

$$\nu(A)\leq\mu(A)\quad\textit{for all measurable subset } A \textit{ of } M,\tag{11}$$

and moreover, there exist positive constants p,ρ,N and L so that (M,d,ν) satisfies the assumptions of Theorem 2.1, then, for every $k\in\mathbb{N}^*$ we have

$$\lambda_k(M,g,\mu)\leq\frac{16N}{\rho^2}+16N(4N^2L)^{2/m}\left(\frac{\mu(M)}{\nu(M)}\right)^{1+2/p}\left(\frac{k}{\mu(M)}\right)^{2/p}.\tag{12}$$

Proof. Take (M,d,ν) as an $m-m$ space. According to Theorem 2.1, for every $2k \in \mathbb{N}^*$, we have a family of $2k$ mutually disjoint capacitors $\{(F_i,G_i)\}_{i=1}^{2k}$ and $2k$ Lipschitz functions f_i's such that for every $1 \le i \le 2k$, $\nu(F_i) \ge \frac{\nu(M)}{4Nk}$ and $\mathrm{dil}_d(f_i)$ satisfies Inequality (8). Here, we have $|\nabla_g f| \le \mathrm{dil}_{d_g}(f)$ almost everywhere. Since $d \le d_g$ and $\nu \le \mu$ we get

$$|\nabla_g f| \le \mathrm{dil}_{d_g}(f) \le \mathrm{dil}_d(f) \le \frac{1}{\rho} + (4N^2L)^{1/p}\left(\frac{2k}{\nu(M)}\right)^{1/p},$$

and

$$\mu(F_i) \ge \nu(F_i) \ge \frac{\nu(M)}{4Nk}.$$

Since the support of f_i's are disjoint and $\sum_{i=1}^{2k}\mu(A_i^r) \le \mu(M)$, at least k of them have measure smaller than $\frac{\mu(M)}{k}$. Up to re-ordering, we assume that for the first k of the A_i^r's we have

$$\mu(A_i^r) \le \frac{\mu(M)}{k}.$$

Therefore,

$$\begin{aligned}\lambda(M,g,\mu) \le \max_i R(f_i) &\le \max_i \left(\frac{1}{\rho} + (4N^2L)^{1/p}\left(\frac{2k}{\nu(M)}\right)^{1/p}\right)^2 \frac{\mu(A_i^r)}{\mu(A_i)} \\ &\le 16N\left(\frac{1}{\rho^2} + (4N^2L)^{2/p}\left(\frac{2k}{\nu(M)}\right)^{2/p}\right)\frac{\mu(M)}{\nu(M)},\end{aligned}$$

and we obtain Inequality (12). □

3. Eigenvalues of Immersed Submanifolds of $\mathbb{R}^N$

In this section, we prove Theorem 1.1. Let S be an m-dimensional immersed submanifold of $\mathbb{R}^{m+p}$, (with or without boundary). We recall that G is the Grassmannian of all m-vector spaces in $\mathbb{R}^{m+p}$ endowed with an invariant Haar measure with total measure 1. We define the *mean intersection index* of S as follows:

$$\bar{\imath}(S) := \int_G i_H(S)dH,$$

where $i_H(S) := \sup_{P\perp H} \sharp(S\cap P)$ and P is an affine p-plane orthogonal to H.

Similarly, for every $r>0$, we define the *r-local intersection index* of S by:

$$\bar{\imath}_r(S) := \sup_{x\in S}\int_G i_H(S\cap B(x,r))dH,$$

where $B(x,r) \subset \mathbb{R}^{m+p}$ is an Euclidean ball of radius r centered at x.

Let $H \in G$ and $\pi_H : S \to H$ be the orthogonal projection of S on H. The following lemma extends what is done in [6, Lemma 2.1].

Lemma 3.1. *Let S be an m-dimensional immersed submanifold of $\mathbb{R}^{m+p}$, (not necessarily without boundary). Then we have*

$$Vol(S) \leq C_m \bar{\imath}(S)\mathrm{Vol}(\pi_{\mathrm{H}}(\mathrm{S})), \tag{13}$$

where C_m is a constant depending only on the dimension m of S.

Proof. Since for almost all $H \in G$, a point in $\pi_H(S)$ has finite number of preimages, one can take a generic H and get

$$\int_S \pi_H^* v_H = \int_S |\theta_H(x)| v_S \leq \int_{\pi_H(S)} i_H(S) v_H = i_H(S)\mathrm{Vol}(\pi_H(S)),$$

where v_S and v_H are volume elements of S and H respectively and

$$|\theta_H(x)| v_S = \pi_H^* v_H.$$

Now, by integrating over G we get

$$\begin{aligned} \int_G i_H(S)\mathrm{Vol}(\pi_{\mathrm{H}}(\mathrm{S}))\mathrm{dH} &\geq \int_G dH \int_S |\theta_H(x)| v_S \\ &= \int_S \left(\int_G |\theta_H(x)| dH \right) v_S \\ &= I(G)\mathrm{Vol}(S), \end{aligned} \tag{14}$$

where $I(G) := \int_G |\theta_H(x)| dH$. The last equality comes from the fact that $I(G)$ does not depend on the point x (see [6, page 101]). We also have

$$\begin{aligned} \int_G i_H(S)\mathrm{Vol}(\pi_{\mathrm{H}}(\mathrm{S}))\mathrm{dH} &\leq \sup_H \mathrm{Vol}(\pi_H(S))\bar{\imath}(S) \\ &\leq 2\mathrm{Vol}(\pi_{H_0}(S))\bar{\imath}(S), \end{aligned} \tag{15}$$

where H_0 is an m-plane such that $2\mathrm{Vol}(\pi_{H_0}(S)) \geq \sup_H \mathrm{Vol}(\pi_H(S))$. By Inequalities (14) and (15), we get the following inequality

$$\mathrm{Vol}(\pi_{H_0}(S)) \geq \frac{I(G)\mathrm{Vol}(S)}{2\bar{\imath}(S)}.$$

This proves Inequality (13) with $C_m = \frac{2}{I(G)}$. □

Let M be an m-dimensional immersed submanifold of $\mathbb{R}^{m+p}$. Throughout the rest of this section, for every $\varepsilon \geq 0$, M_ε^D stands for $M \setminus D$, where D is any open subdomain of M such that $M \setminus D$ is a smooth manifold with the smooth boundary and $\mathrm{Vol}(D) = \varepsilon \mathrm{Vol}(M)$.

Corollary 3.1. *For all $x \in \mathbb{R}^{m+p}$ and $\varepsilon \geq 0$, we have*

$$Vol\big(M_\varepsilon^D \cap B(x,s)\big) \leq \frac{2\, Vol(B^m)}{I(G)} \bar{\imath}_r(M_\varepsilon^D) s^m, \quad \forall\ 0 < s \leq r; \tag{16}$$

$$Vol\big(M_\varepsilon^D \cap B(x,r)\big) \leq \frac{2\, Vol(B^m)}{I(G)} \bar{\imath}(M_\varepsilon^D) r^m, \quad \forall r > 0, \tag{17}$$

where B^m is the m-dimensional Euclidean unit ball.

Proof. Replacing S by $M_\varepsilon^D \cap B(x,s)$ in Lemma 3.1, we obtain

$$\begin{aligned}\mathrm{Vol}\left(M_\varepsilon^D \cap B(x,s)\right) &\leq \frac{2}{I(G)}\bar{\imath}\left(M_\varepsilon^D \cap B(x,s)\right)\mathrm{Vol}\left(\pi_H\left(M_\varepsilon^D \cap B(x,s)\right)\right)\\ &\leq \frac{2\mathrm{Vol}(B^m)}{I(G)}\bar{\imath}_s\left(M_\varepsilon^D\right)s^m,\end{aligned}$$

where B^m is the m-dimensional Euclidean unit ball.
The last inequality comes from

$$\mathrm{Vol}\left(\pi_{H_0}\left(M_\varepsilon^D \cap B(x,s)\right)\right) \leq \mathrm{Vol}\left(\pi_{H_0}\left(B(x,s)\right)\right) \leq \mathrm{Vol}(B^m)s^m$$

Since $\bar{\imath}_s\left(M_\varepsilon^D\right) \leq \bar{\imath}_r\left(M_\varepsilon^D\right)$ for all $0 < s \leq r$ and $\bar{\imath}_s\left(M_\varepsilon^D\right) \leq \bar{\imath}\left(M_\varepsilon^D\right)$ for all $s > 0$, therefore, we derive Inequalities (16) and (17). □

Remark 3.1. *For $\varepsilon = 0$, we have $M_\varepsilon^D = M$. Hence, we have the Inequalities (16) and (17) for M_ε^D replaced by M.*

Proof of Theorem 1.1. This theorem is a straightforward consequence of Corollary 2.1. We begin with giving candidates for a distance d and a measure μ such that the assumptions of Corollary 2.1 are satisfied. Let $d = d_{eu}$ be the Euclidean distance in $\mathbb{R}^{m+p}$ and $\mu = \mu_\epsilon^D$ where $\mu_\epsilon^D(A)$ is the Riemannian volume of $A \cap M_\varepsilon^D$. One can easily check that (M, d_{eu}) has the $(2,N)$-covering property where N depends only on the dimension of the ambient space $\mathbb{R}^{m+p}$. Moreover, one can consider it as a function depending only on the dimension m (see [6, page 106]). There also exists $L > 0$ such that $\mu_\varepsilon^D(B(x,s)) \leq Ls^m$ for $s \leq \rho$. Indeed, we consider two cases:

- Take $\rho = r$. According to Corollary 3.1, one can take $L = \frac{2\mathrm{Vol}(B^m)}{I(G)}\bar{\imath}_r(M_\varepsilon^D)$. Therefore, Corollary 2.1 implies

$$\lambda_k(M) \leq \alpha_m \frac{1}{r^2} + \beta_m \frac{\bar{\imath}_r(M_\varepsilon^D)^{2/m}}{(1-\varepsilon)^{1+2/m}}\left(\frac{k}{\mathrm{Vol}(M)}\right)^{2/m}. \tag{18}$$

- Take $\rho = \infty$. According to Corollary 3.1, one can take $L = \frac{2\mathrm{Vol}(B^m)}{I(G)}\bar{\imath}(M_\varepsilon^D)$. Therefore, Corollary 2.1 implies

$$\lambda_k(M) \leq \beta_m \frac{\bar{\imath}(M_\varepsilon^D)^{2/m}}{(1-\varepsilon)^{1+2/m}}\left(\frac{k}{\mathrm{Vol}(M)}\right)^{2/m}. \tag{19}$$

The left hand-sides of Inequalities (18) and (19) do not depend on D. Hence, taking the infimum over D, we get Inequalities (3) and (4). □

4. Eigenvalues of Complex Submanifolds of $\mathbb{C}P^N$

In this section, we provide the proof of Theorem 1.2. Before going into the proof we need to recall the universal inequality proved by El Soufi, Harrell and Ilias which is the key idea of the proof. The following lemma is a special case of that universal inequality [9, Theorem 3.1] (see also [5]):

Lemma 4.1. *Let M^m be a compact complex manifold of complex dimension m and $\phi : M \to \mathbb{C}P^N$ be a holomorphic immersion. Then the eigenvalues of the Laplace-Beltrami operator on $(M, \phi^* g_{FS})$ satisfy the following inequality:*

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{2}{m} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)(\lambda_i + c_m), \tag{20}$$

where $c_m = 2m(m+1)$.

Another useful result is the following recursion formula given by Cheng and Yang:

Lemma 4.2. *([4, Corollary 2.1]) If a positive sequence of numbers $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{k+1}$, satisfies the following inequality*

$$\sum_{i=1}^{k} (\mu_{k+1} - \mu_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} \mu_i (\mu_{k+1} - \mu_i), \tag{21}$$

then

$$\mu_{k+1} \leq (1 + \frac{4}{n}) k^{2/n} \mu_1.$$

Theorem 4.1. *Let M^m be a compact complex manifold of complex dimension m admitting a holomorphic immersion $\phi : M \to \mathbb{C}P^N$. Then for every $k \in \mathbb{N}^*$ we have*

$$\lambda_{k+1}(M, \phi^* g_{FS}) \leq 2(m+1)(m+2)k^{\frac{1}{m}} - 2m(m+1). \tag{22}$$

Proof of Theorem 4.1. According to Lemma 4.1, the eigenvalues of the Laplace operator on M satisfy universal Inequality (20). We replace λ_i by $\mu_i := \lambda_i + c_m$ in Inequality (20) and we obtain:

$$\sum_{i=1}^{k} (\mu_{k+1} - \mu_i)^2 \leq \frac{2}{m} \sum_{i=1}^{k} \mu_i (\mu_{k+1} - \mu_i).$$

One now has a positive sequence of numbers $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{k+1}$ that satisfies Inequality (21) with $n = 2m$. Applying the recursion formula of Cheng and Yang, we get the following inequality:

$$\mu_{k+1} \leq (1 + \frac{4}{2m}) k^{2/2m} \mu_1. \tag{23}$$

By replacing μ_i by $\lambda_i + c_m$ in Inequality (23), we obtain:

$$\lambda_{k+1}(M, \phi^* g_{FS}) \leq (1 + \frac{2}{m})(\lambda_1(M, \phi^* g_{FS}) + c_m) k^{1/m} - c_m.$$

Since M is a compact manifold, $\lambda_1(M, \phi^*(g_{FS}) = 0$. Therefore,

$$\lambda_{k+1}(M, \phi^* g_{FS}) \leq (1+\frac{2}{m}) c_m k^{1/m} - c_m = 2(m+1)(m+2)k^{1/m} - 2m(m+1),$$

which completes the proof. $\square$

As we mentioned in the introduction, for $k = 1$ we get a sharp upper bound:

$$\lambda_2(M, \phi^* g_{FS}) \leq \lambda_2(\mathbb{C}P^m, g_{FS}) = 4(m+1). \tag{24}$$

In [2], Bourguignon, Li and Yau obtained an upper bound for the first non-zero eigenvalue of a complex manifold (M, ω) which admits a *full* holomorphic immersion (i.e. $\Phi(M)$ is not contained in any hyperplane of $\mathbb{C}P^N$) into $\mathbb{C}P^N$ as following:

$$\lambda_2(M, \omega) \leq 4m \frac{N+1}{N} d([\Phi], [\omega]). \tag{25}$$

Here, $d([\Phi], [\omega])$ is the *holomorphic immersion degree* – a homological invariant – defined by

$$d([\Phi], [\omega]) = \frac{\int_M \Phi^*(\omega_{FS}) \wedge \omega^{m-1}}{\int_M \omega^m},$$

where ω_{FS} is the Kähler form of $\mathbb{C}P^N$ with respect to the Fubini-Study metric and ω is Kähler form on M.
If one takes $\omega = \Phi^*(\omega_{FS})$, then $d([\Phi], [\omega]) = 1$ and we get Inequality (24) as a corollary of Inequality (25). Theorem 4.1 gives us another proof without assuming the immersion to be full.

For any full holomorphic immersion Φ of the surface Σ_γ with genus γ inequality 25 gives

$$\lambda_2(\Sigma_\gamma, \omega) \mathrm{Vol}(\Sigma_\gamma, \omega) \leq 4 \frac{N+1}{N} \deg(\Phi(\Sigma_\gamma)).$$

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